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Recovering velocity from temperature measurements in buoyancy-driven flow

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Data Assimilation

- Optimally combine a mathematical model with data
- Assumes access to measurements continuously in time
- Typically measurements are not comprehensive or only cover large scales
- Typical problem construction[1]:
  - $u(t)$: state of a dynamical system
  - $I_h(u(t))$: observations of system at a coarse spatial resolution (size h)
  - Observations $\{I_h(u(t)) : t \in [0,T]\}$ are known over an interval of time
  - Goal: **construct an increasingly accurate initial condition** from which predictions of $u(t)$ can be made for $t > T$

[1] Azouani, Olson, Titi, "Continuous Data Assimilation using General Interpolant Observables" (2014)

Common Approaches

- Kalman filter and its variants (extended Kalman filter, ensemble Kalman filter)
- 4DVar (PDE-constrained optimization using adjoint equations)
- PINNs
- Direct discrete loss optimization (fields parametrized by grid values not NNs)[2]
- Nudging (Newtonian relaxation)[1]

Each of these approaches has a different mix of hard and soft constraints with optimization

[1]Azouani, Olson, Titi, "Continuous Data Assimilation using General Interpolant Observables" (2014)
[2]Karnakov, Litvinov, Koumoutsakos, "Solving inverse problems in physics by optimizing a discrete loss" (2024)

Applications and Extensions

- The canonical application for data assimilation for weather forecasting
- Many extensions of the standard problem exist. In addition to recovering the initial condition, one may want to
  - Recover missing model parameters
  - Estimate parameters for nonphysical modeling terms like turbulence closures
  - Recover unknown boundary conditions
  - Estimate initial conditions for state variables lacking measurement data

Problem Motivation

- "Weather in a room" problem: small-scale inference of flow in a mixed convection setting
  - Flow is driven both by buoyancy from temperature differences and from an inlet (HVAC system)
- For practical use in small-scale settings, velocity measuring devices are much more expensive and unreliable than temperature sensors
- Problem: can we infer velocity (and full temperature state) from temperature measurements alone?[1,2,3]
- Similar to well-known "Charney conjecture[1]" can the state of the atmosphere be reconstructed using only temperature measurements?

[1]Farhat, Lunasin, and Titi, "On the Charney Conjecture of Data Assimilation" (2016)
[2]Farhat, Lunasin, and Titi, "Data assimilation for convection in porous media employing only temperature" (2016)
[3]Agasthya, Clark Di Leoni, and Biferale, "Reconstructing Rayleigh–Bénard flows out of temperature-only measurements using nudging" (2022)

Mathematical Model of 2D Convection

- On a domain $\Omega \subseteq \mathbb{R}^2$, let $\mathbf{u}$ be a fluid's velocity and θ its temperature. We use the Boussinesq approximation for buoyancy-driven incompressible flow:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \Delta \mathbf{u} + g\alpha(\theta - \theta_0)\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta$$

- Boundary conditions are also needed

Nondimensionalization

- Rescaling using the characteristic scales

$$[L] = h, \qquad [t] = h^2/\nu, \qquad [\theta] = \delta\theta, \qquad [M] = \rho_0 h^3$$

and defining

$$\mathrm{Pr} = \nu/\kappa, \qquad \mathrm{Ra} = g\alpha\partial\theta h^3/(\nu\kappa)$$

yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u} + \mathrm{Ra}\,\theta\hat{\mathbf{z}}$$

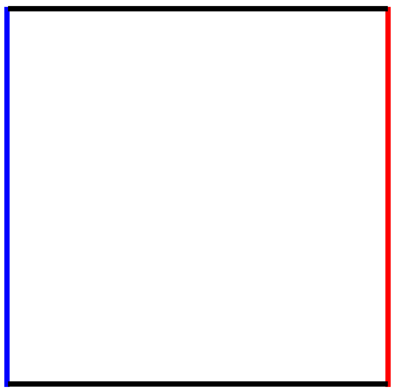
$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = \kappa\frac{1}{\mathrm{Pr}}\Delta\theta$$

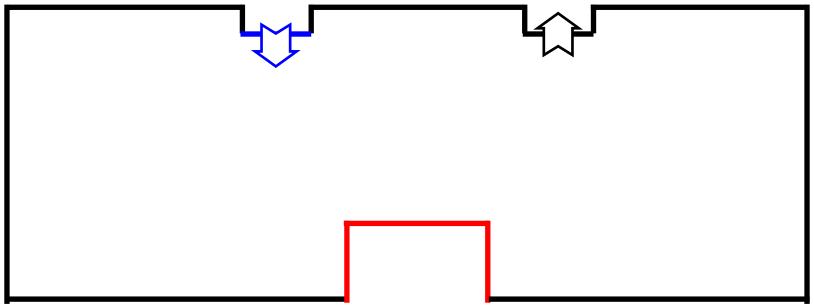
Boundary conditions

- For velocity:
  - No-slip boundary conditions ($\mathbf{u} = \mathbf{0}$)
  - Prescribed Dirichlet conditions
  - Free boundary ($p = 0$)
- For temperature:
  - Prescribed Dirichlet boundary conditions
  - Adiabatic (Neumann) boundary conditions ($\hat{\mathbf{n}} \cdot \nabla\theta = 0$)

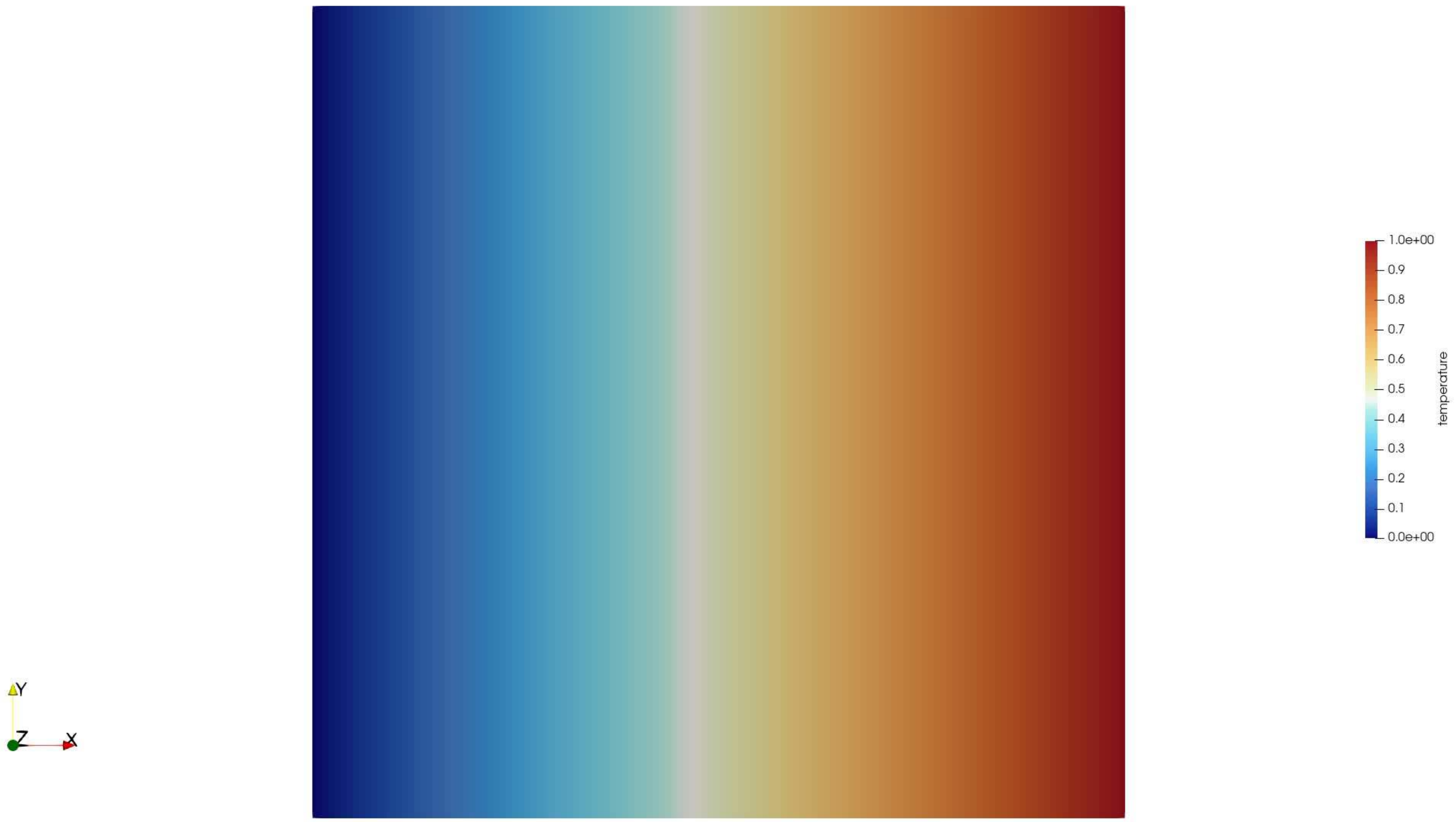
Test Problem Geometries



Box (heated and cooled on sides)
- No-slip BCs for velocity on all edges
- Adiabatic temperature BCs on all insulated surfaces



Room (cool air enters via vent, object in room heats air)
- No-slip BCs for velocity except at inlet (prescribed) and outlet (zero pressure)
- Adiabatic temperature BCs on all insulated surfaces



1.0e+00
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0e+00
temperature
Y
Z
X

2.5e+01
20
15
10
5
0
-2.0e+00
temperature
Y
Z
X

Problem Statement:

Given the following:

- observations $\{I_h(\theta(t)) : t \in [0,T]\}$ of coarse-scale temperature continuously over an interval,
- knowledge of velocity boundary conditions,
- knowledge that $\mathbf{u}(t), \theta(t), t \in [0,T]$ are governed by the Boussinesq equations on $\Omega \subseteq \mathbb{R}^2$,

construct an approximate state $(\mathbf{u}(T), \theta(T))$ for velocity and temperature, which could be used to make forecasts for $t > T$.

Velocity information other than boundary conditions is unknown.

Nudging Framework

- [1]Working in the classic data assimilation problem framework, suppose we have a dynamical system

$$u_t = F(u)$$

- "Nudging" or Newtonian Relaxation works by simulating a parallel assimilating system for $\tilde{u}(t)$ using measurements $\{I_h(u(t)) : t \in [0,T]\}$:

$$\tilde{u}_t = F(\tilde{u}) + \mu(I_h(u) - I_h(\tilde{u}))$$

- Under appropriate conditions (e.g. dissipativity), $\tilde{u} \to u$
- Key difference with our problem: the measurements are for the full state u, whereas we only have temperature measurements but are trying to recover temperature and velocity

[1]Azouani, Olson, Titi, "Continuous Data Assimilation using General Interpolant Observables" (2014)

Nudging Theory

Azouani, Olson, and Titi (2014) proved that

- if $\mathbf{u}$ is governed by the incompressible Navier-Stokes equations with periodic or no-slip boundary conditions,
- and there exists a C such that

$$\|\varphi - I_h(\varphi)\|^2_{L^2(\Omega)} \leq Ch^2\|\varphi\|^2_{H^1(\Omega)} \qquad \text{for all } \varphi \in H^1(\Omega),$$

then for small enough h, the assimilating velocity $\tilde{\mathbf{u}}$ converges to $\mathbf{u}$ exponentially as $t \to \infty$ in $L^2(\Omega)$.

- Similar results have been proven for other dissipative dynamical systems
- Analyzing the theory for the velocity inference problem is an intriguing area for future research

[1]Azouani, Olson, Titi, "Continuous Data Assimilation using General Interpolant Observables" (2014)

Previous work

- Farhat et al. worked on temperature-nudging in the context of modified equations (namely, geostrophic flow and porous media)
- Agasthya et al. recently studied convergence of temperature-only nudging with different levels of density in Rayleigh–Bénard convection
  - Here the geometry is simple: heat on bottom, cool on top
- Our problem geometry is more complicated and we intend to study a mixed convection setting (where the convection is driven by inflow as well as buoyancy)
- We also intend to utilize nonlinear nudging[4]

[1]Farhat, Lunasin, and Titi, "On the Charney Conjecture of Data Assimilation" (2016)
[2]Farhat, Lunasin, and Titi, "Data assimilation for convection in porous media employing only temperature" (2016)
[3]Agasthya, Clark Di Leoni, and Biferale, "Reconstructing Rayleigh–Bénard flows out of temperature-only measurements using nudging" (2022)
[4]Carlson, Larios, Titi, "Super-Exponential Convergence Rate of a Nonlinear Continuous Data Assimilation Algorithm" (2022)

Computational Methods

- Use finite elements because of flexibility to deal with complex geometries and different meshes
  - Finite element interpolation satisfies the interpolant condition from AOT
- Unstructured triangular mesh
- P_2/P_1 mixed element (Taylor-Hood element) used for velocity and pressure
- P_2 element used for temperature
- Crank-Nicolson for time-stepping
- Velocity is initialized from zero, and temperature is initialized using interpolation from sensor values

Full Equations

Original (true) system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \Delta \mathbf{u} + g\alpha\theta\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla\theta = \kappa\Delta\theta$$

$$\mathbf{v} - \lambda^2\Delta\mathbf{v} = \mathbf{u}$$

Assimilating (nudged) system

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{u}} = -\frac{1}{\rho_0}\nabla \tilde{p} + \nu \Delta \tilde{\mathbf{u}} + g\alpha\tilde{\theta}\hat{\mathbf{z}}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla\tilde{\theta} = \kappa\Delta\tilde{\theta} + \mu(I_h(\tilde{\theta}) - I_h(\theta))$$

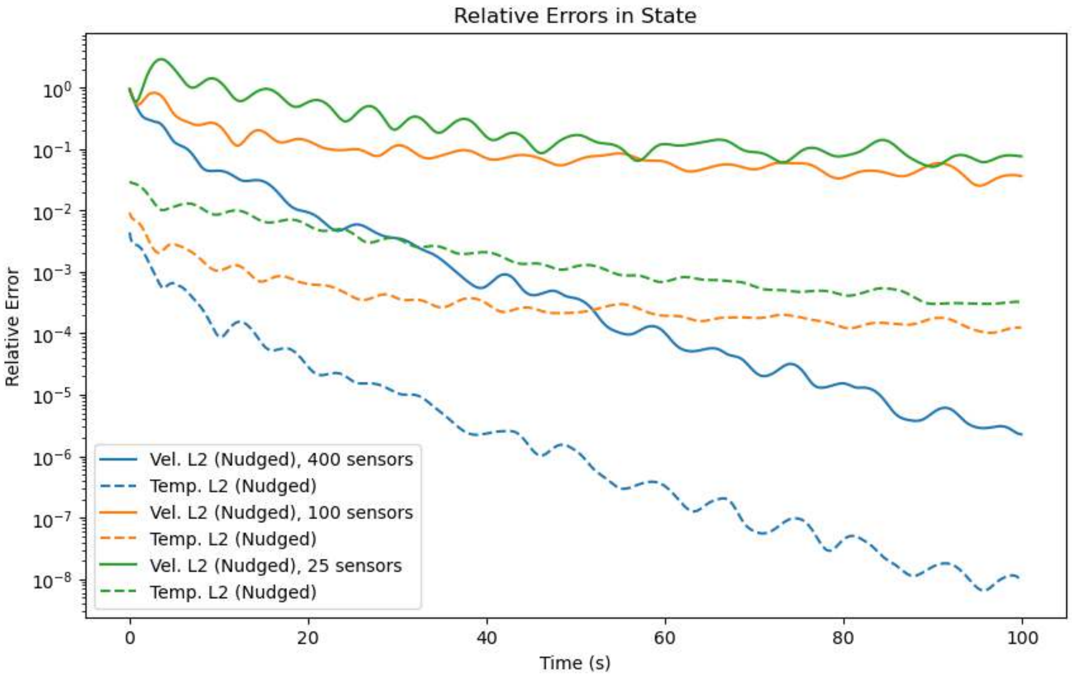
$$\tilde{\mathbf{v}} - \lambda^2\Delta\tilde{\mathbf{v}} = \tilde{\mathbf{u}}$$

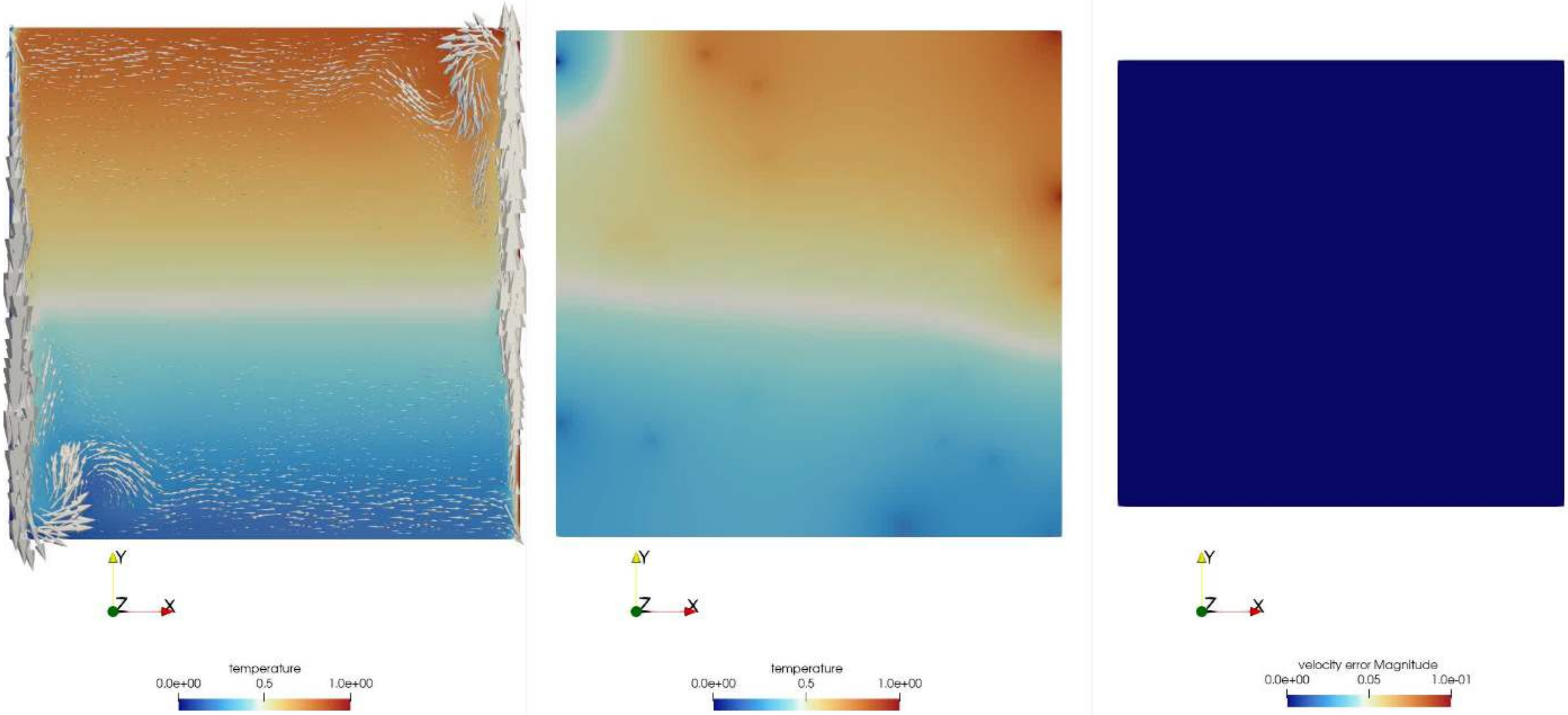
The last equation represents the Leray-α turbulence closure model[1], which helped me speed up the computations by using a relatively coarse mesh.

[1]Bisa, “The Leray-α Model of Turbulence” (2014)

Convergence on Box Problem

- Rayleigh number is 10^8
- Chosen because turbulence is present but still relatively inexpensive to simulate on a coarse grid
- Sensors are chosen uniformly from a grid of 5405 vertices





Y
Z
X
temperature
0.0e+00
0.5
1.0e+00
Y
Z
X
temperature
0.0e+00
0.5
1.0e+00
Y
Z
X
velocity error Magnitude
0.0e+00
0.05
1.0e-01

Nonlinear nudging

- Idea: modify the nudging term to be nonlinear

$$\frac{\partial\tilde{\theta}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla\tilde{\theta} = \kappa\Delta\tilde{\theta} + \mu\mathcal{N}(I_h(\tilde{\theta}) - I_h(\theta))$$

where

$$\mathcal{N}(\varphi) = \frac{\varphi}{\|\varphi\|_{L^2}^{\gamma}}, \qquad \gamma \in [0,1).$$

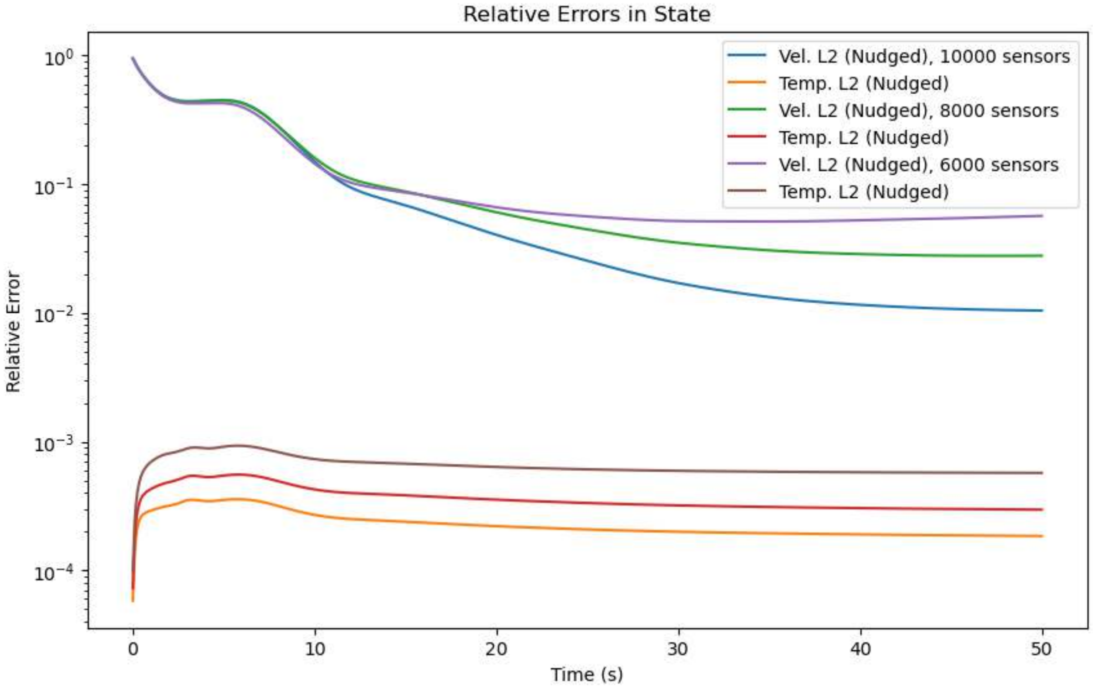
$\gamma = 0$ recovers the linear nudging case while $\gamma > 0$ effectively increases the strength of the nudging as $\tilde{\theta} \to \theta$.

- Larios et al. proved that nonlinear nudging achieves super-exponential convergence up to a threshold[1]

[1] Carlson, Larios, Titi, "Super-Exponential Convergence Rate of a Nonlinear Continuous Data Assimilation Algorithm" (2022)

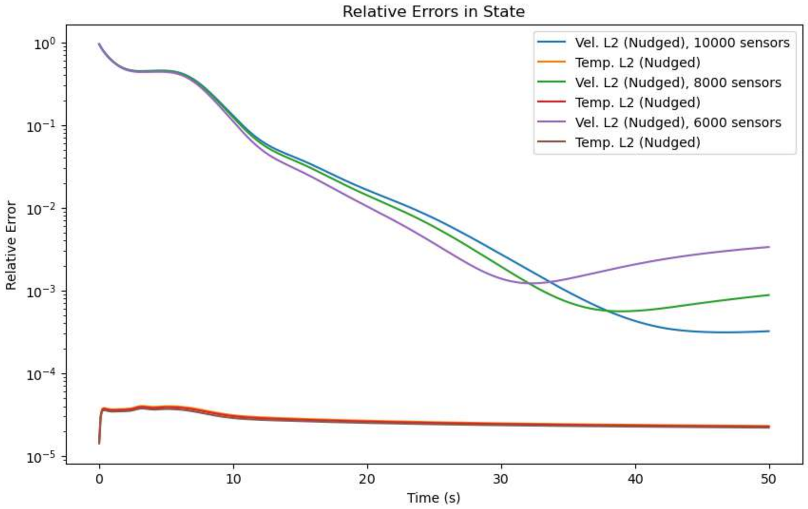
Convergence on Room Problem

Rayleigh number is 2.16×10^4



Convergence on Room Problem, cont.

This experiment nudges uses an interpolated state, not just at the sensor locations



Future work and takeaways

- An interesting and important extension of this problem is to recover the velocity boundary conditions as well (especially at the inlet)
- Theoretical analysis of the convergence of nudging for this problem seems doable and important
- I would like to compare nudging with other methods for solving this problem, including discrete loss optimization or 4DVar
- Performing a more detailed study of temperature sensor placement seems relevant

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Thank You