# Collinear Equilibria in a Simplified Four-Body Problem

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# Outline

Gravitational $N$-Body Problems

The Simplified Four-Body Problem

Collinear Equilibria

Stability

Future Work

# Gravitational $N$-Body Problems

- A system of differential equations modeling the motion of $N$ masses under their mutual gravitational attraction
- Studied since Newton
- Used by NASA to design trajectories
- $N > 2$: No closed-form solution

# Example: The Kepler Problem ($N = 2$)

Let $\mathrm{x}_1, \mathrm{x}_2 : \mathbb{R} \to \mathbb{R}^3$ represent the position of masses $m_1$ and $m_2$, respectively. The equations of motions under gravitational attraction are

$$m_1 \frac{d^2\mathrm{x}_1}{dt^2} = -\frac{Gm_1 m_2}{\|\mathrm{x}_2 - \mathrm{x}_1\|^3}(\mathrm{x}_2 - \mathrm{x}_1), \qquad m_2 \frac{d^2\mathrm{x}_2}{dt^2} = -\frac{Gm_1 m_2}{\|\mathrm{x}_1 - \mathrm{x}_2\|^3}(\mathrm{x}_1 - \mathrm{x}_2).$$

- Model any two-body system
- Solvable for any initial conditions
- Solutions have a simple form

# $N$-Body Problems for $N > 2$

- $N > 2$: No closed-form solution
- Idea: Impose additional simplifications or symmetries
- Example: The Restricted 3-body Problem
  - $N = 3$, but we let one of the masses be zero so that it does not affect the motion of the others
  - A great model for spacecraft or satellites

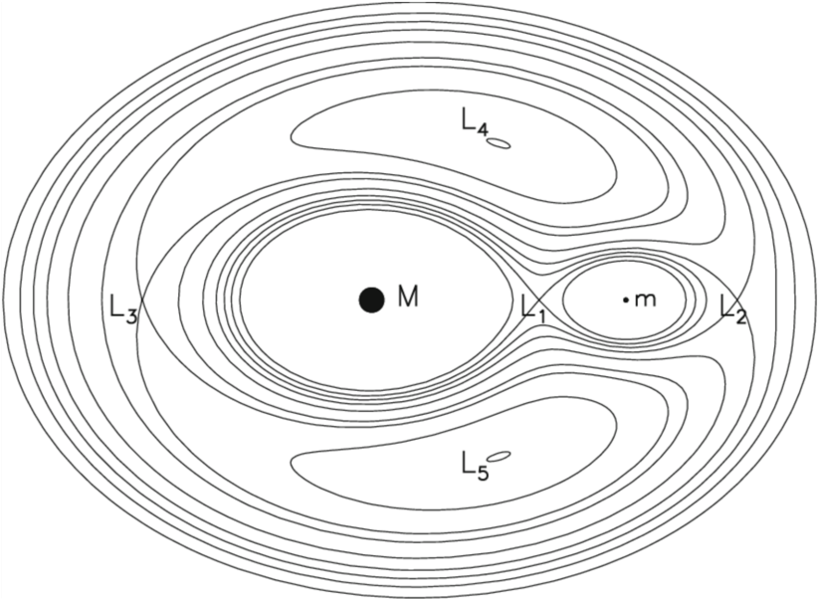


Figure: The equipotential surfaces and equilibrium points in the circular restricted 3-body problem

# A "Bridge" Problem

- Two primaries $M_1, M_2$ and two small masses $m_1, m_2$
- Assumptions: $m_1, m_2 \ll M_1, M_2$, and $M_1, M_2$ move on Kepler orbits
- "Between" the restricted 3-body problem and the 4-body problem
- Which characteristics does it share with each of these problems?

# How to Find Solutions

- Impose some symmetry or look for a special type of solution
- Look for equilibrium points in rotating coordinates
- Look for collinear equilibria

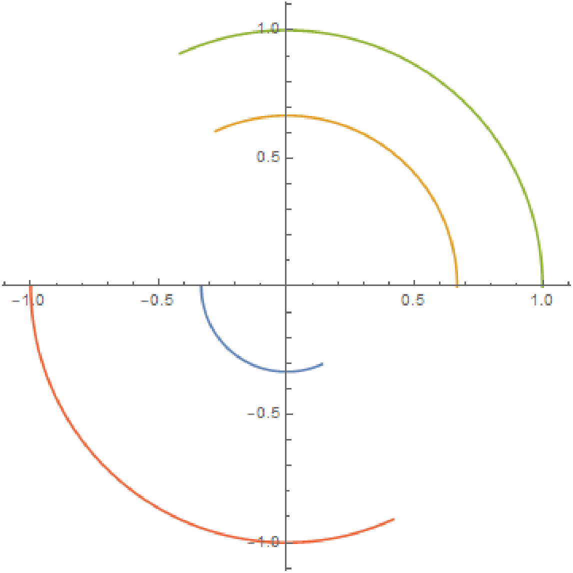


Figure: A collinear equilibrium orbit in the simplified 4-body problem.

# Finding Collinear Equilibria

- The general four-body problem has twelve collinear equilibria
- Do they persist in the simplified problem?
- Imposing this constraint yields

$$0 = \frac{Mu_1}{a^3} - \frac{m_2(u_1 - u_2)}{|u_1 - u_2|^3} - \frac{M_1(u_1 + aM_2/M)}{|u_1 + aM_2/M|^3} - \frac{M_2(u_1 - aM_1/M)}{|u_1 - aM_1/M|^3}$$

$$0 = \frac{Mu_2}{a^3} + \frac{m_1(u_1 - u_2)}{|u_1 - u_2|^3} - \frac{M_1(u_2 + aM_2/M)}{|u_2 + aM_2/M|^3} - \frac{M_2(u_2 - aM_1/M)}{|u_2 - aM_1/M|^3}$$

# The Solution

- Create dimensionless parameters $z_i = u_i/a$ and $\mu = m_1/(m_1 + m_2)$ to make the equations simpler:

$$z_1 = \frac{m_2}{M}\frac{\operatorname{sign}(z_1 - z_2)}{(z_1 - z_2)^2} + \frac{\mu \operatorname{sign}(z_1 + 1 - \mu)}{(z_1 + 1 - \mu)^2} + \frac{(1-\mu)\operatorname{sign}(z_1 - \mu)}{(z_1 - \mu)^2}$$

$$z_2 = \frac{m_1}{M}\frac{\operatorname{sign}(z_2 - z_1)}{(z_1 - z_2)^2} + \frac{\mu \operatorname{sign}(z_2 + 1 - \mu)}{(z_2 + 1 - \mu)^2} + \frac{(1-\mu)\operatorname{sign}(z_2 - \mu)}{(z_2 - \mu)^2}$$

- Prove separately in each of the twelve regions

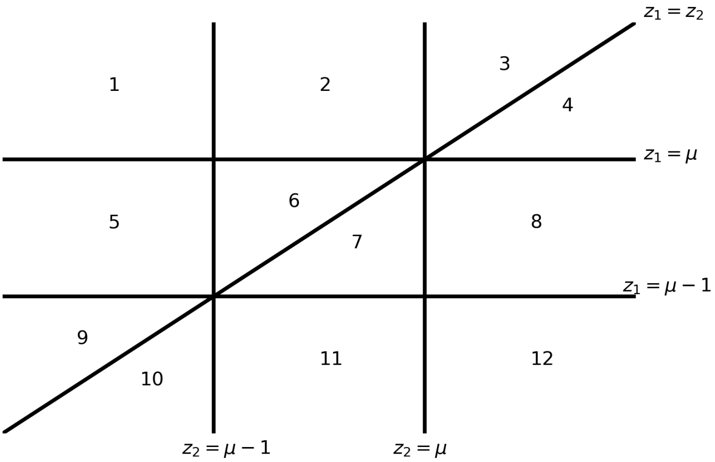


Figure: The twelve regions of the $z_1 z_2$ plane corresponding to different ordering with respect to the two primaries.

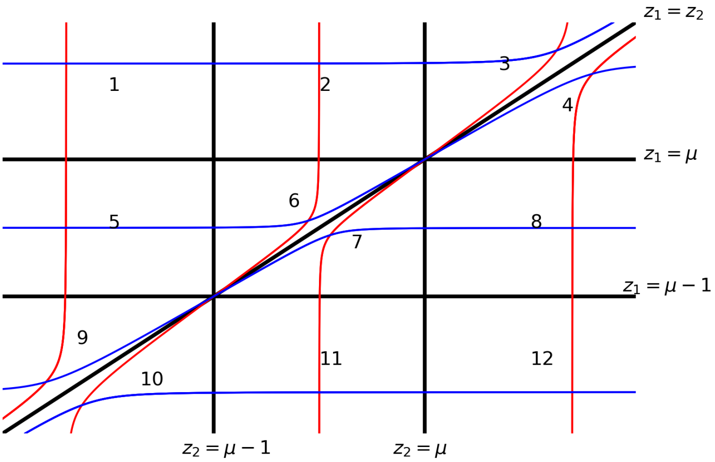


Figure: The contours representing the two equations in the twelve regions of the $z_1 z_2$ plane.

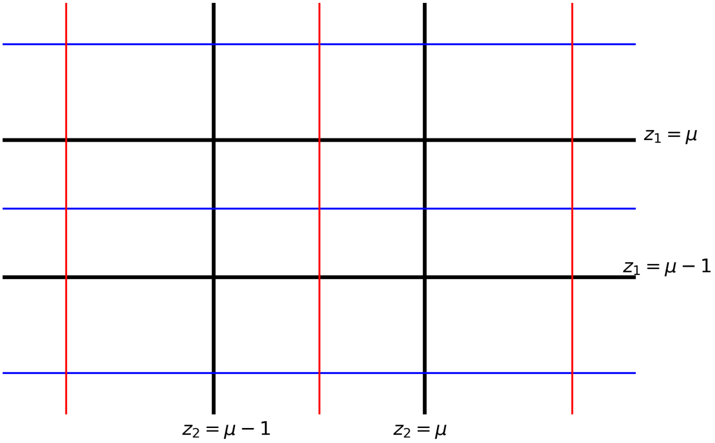


Figure: The contours representing the results of two "stacked" 3-body problems.

# Stability

- Our system is $\dot{z} = J\nabla H(z)$
- If $z^*$ is an equilibrium point, so that $\nabla H(z^*) = 0^T$, then expand the solutions of the sin a Taylor series around $z^*$:

$$z - z^* = J\nabla H(z^*)(z - z^*) + \frac{1}{2}(z - z^*)^T J D^2 H(z^*)(z - z^*)$$

- Stability of an equilibrium point is determined by the eigenvalues of the Hessian matrix

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{1-\mu} & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\mu} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\mu} \\
-2b_{22} & 0 & 0 & -2b_{25} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & b_{22} & 0 & 0 & b_{25} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{22} & 0 & 0 & b_{25} & 0 & 0 & 0 & 0 & 0 & 0 \\
-2b_{25} & 0 & 0 & -2b_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & b_{25} & 0 & 0 & b_{55} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & b_{25} & 0 & 0 & b_{55} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$b_{22} = \frac{\partial^2 U}{\partial v_1^2} = -\frac{m_r/M}{|u_1 - u_2|^3} - \frac{\mu\mathcal{M}}{|u_1 + 1 - \mu|^3} - \frac{\mathcal{M}(1-\mu)}{|u_1 - \mu|^3}$$

$$b_{55} = -\frac{\partial^2 U}{\partial v_2^2} = -\frac{m_r/M}{|u_1 - u_2|^3} - \frac{(1-\mathcal{M})\mu}{|u_1 + 1 - \mu|^3} - \frac{(1-\mu)(1-\mathcal{M})}{|u_1 - \mu|^3}$$

$$b_{25} = \frac{\partial^2 U}{\partial v_1 \partial v_2} = \frac{m_r/M}{|u_1 - u_2|^3}$$

# Characteristic Polynomial

- Let
$$b = \frac{\mu}{|u_1 + 1 - \mu|^3} + \frac{(1-\mu)}{|u_1 - \mu|^3} > 0$$
$$c = \frac{b_{25}}{\mathcal{M}(1-\mathcal{M})} = \frac{(m_1 + m_2)/M}{|u_1 - u_2|^3} > 0$$
- Characteristic polynomial factors as the following:

$$\begin{aligned} p(z) = {} & (b + z^2)(b + c + z^2) \\ & ((-1 - b + 2b^2) + (b-2)z^2 - z^4) \\ & ((-1 - (b+c) + 2(b+c)^2) + (b+c-2)z^2 - z^4) \end{aligned}$$

# Eigenvalues

- Eigenvalues change based on values of parameters $b$ and $c$
- Two pairs of pure imaginary roots
- Quartics have more complicated structure:

| $b$ | Root structure |
|---|---|
| $b < 8/9$ | four complex roots |
| $b = 8/9$ | one pair of imaginary roots |
| $8/9 < b < 1$ | two pairs of imaginary roots |
| $b = 1$ | zero and one pair of imaginary roots |
| $b > 1$ | one pair of real roots and one pair of imaginary roots |

# Future Work

- Analyze stable and unstable manifolds
- Prove the existence of certain planar configurations
- Include elliptical orbits for the primaries

# Bibliography


- Klement, Rainer J.
  https://www.researchgate.net/figure/Contours-of-constant-effective-potential-in-the-restricted-three-body-problem_fig1_50909681