Stability of Stochastic Switched Systems

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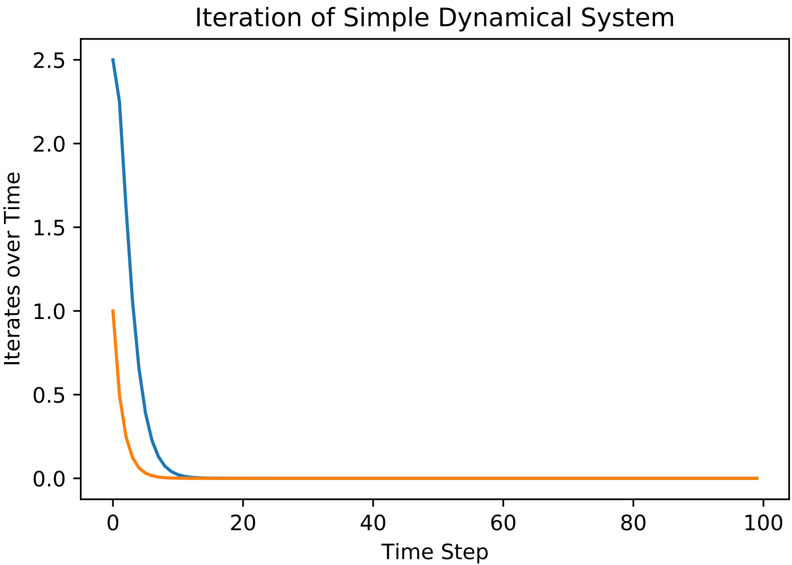
What is a stochastic switched system?

- Dynamical system
- Add two levels of complexity
- Switching
- Stochastic switching

$$M_1 = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}$$

A simple dynamical system



Why should we study switched systems?

- Phenomena with seasonal dependence
- Phenomena with multiple "modes"
- Example: ecology
- Example: server handling traffic

First-Mean Stability

Definition We say that a stochastic switched system (M, X, μ) is *exponentially stable in first mean* to a fixed point $\tilde{x}$ if there exist real constants $C > 0$ and $\beta > 0$ such that for any initial condition x^0 we have

$$\mathbb{E}\left[\|x^k - \tilde{x}\|\right] \leq Ce^{-\beta k}\|x^0 - \tilde{x}\|.$$

Stability

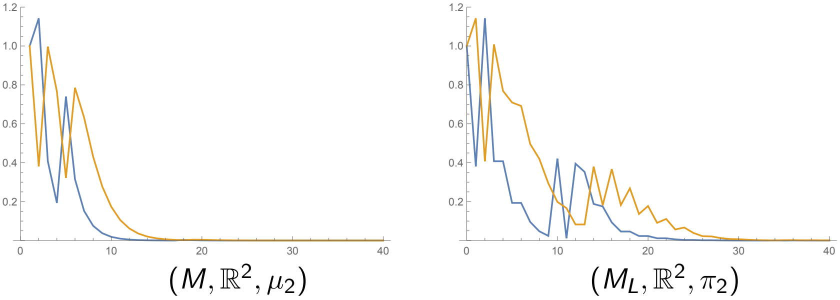


Figure: The undelayed dynamics of the system $(M, \mathbb{R}^2, \mu_2)$ is shown left. The delayed version of this system $(M_L, \mathbb{R}^2, \pi_2)$ for $L = 1$ is shown right. Both are stable.

1 - Radius

- Generalization of the spectral radius

Definition For a linear stochastic switched system $(S, \mathbb{R}^n, \mu)$ with transition operators $\{A_1, A_2, \ldots\}$, the *generalized joint spectral radius* (one-radius) of $(S, \mathbb{R}^n, \mu)$ is defined by

$$\rho_{1,\mu} = \lim_{k \to \infty} (\mathbb{E}[\|A_k \cdots A_1\|])^{1/k}.$$

A simple example

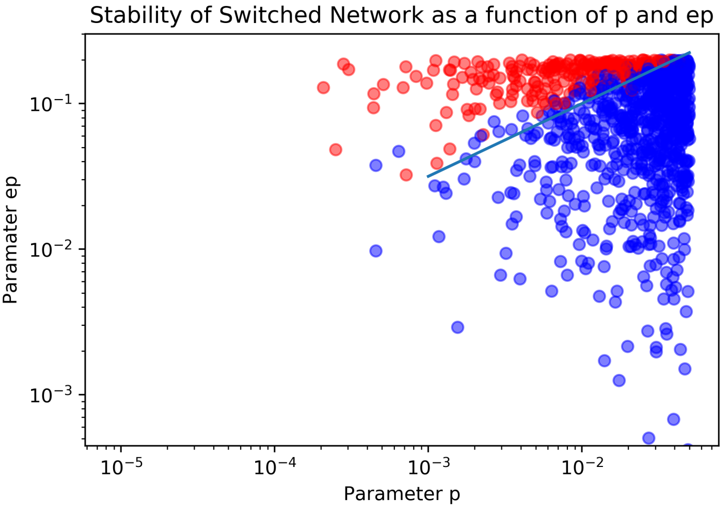
Consider the following switched system $\{M_1, M_2\}$ on $\mathbb{R}^2$ with

$$M_1 = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \epsilon & 0 \\ 1 & \epsilon \end{pmatrix}$$

where we switch according to the Markov chain with transition matrix

$$Q = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

A simple example



A simple example

It turns out that the system is first-mean stable exactly when

$$p > \frac{\epsilon^2}{1 - 2\epsilon + \epsilon^2}$$

Dealing with Nonlinear Systems

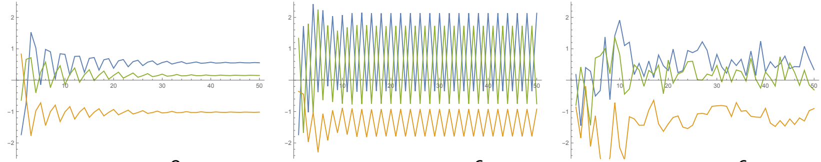
- Linearize the system by taking Lipschitz matrices
- Apply the same results

Time-Delayed Systems

- Interactions may take more than one time step to propagate
- Useful for modeling real-world systems
- We need to know whether or not systems are stable under time-delays

Main result

Theorem Let (M, X, μ) be a stochastic switched system where the support of μ is bounded and each $F \in M$ shares a fixed point $\tilde{x} \in X$. If there is a Lipschitz system $(S, \mathbb{R}^n, \mu')$ of (M, X, μ) that is exponentially stable in first-mean, or if there is a delayed version (M_L, X_L, μ) with corresponding Lipschitz system $(S_L, \mathbb{R}^{(L(n+1)}, \pi')$ that is exponentially stable in first-mean, then (M, X, μ) is patiently first-mean stable.



$(H, \mathbb{R}^3)$ $(H_D, \mathbb{R}^6)$ $(H_L, \mathbb{R}^6, \pi)$

Figure: The stable dynamics of the system $(H, \mathbb{R}^3)$ is shown left. Adding the constant time delays D results in the system $(H_D, \mathbb{R}^6)$ with periodic dynamics, shown center. Adding stochastically time-varying time-delays of size no larger than $L = 1$ similarly destabilizes the system's dynamics. This is shown right for the system $(H_L, \mathbb{R}^6, \pi)$.

Thank you!