Data Assimilation and Parameter Recovery for Rayleigh-Bénard Convection

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What is data assimilation?

- Optimally combine a mathematical model with data
- Use data over time and a model to infer system state
- Example: weather forecasting

Nudging

- Governing system:

$$u_t + F(u) = 0$$

- Partial observations of state: Pu
- "Nudged" system:

$$u_t + F(v) = P(u - v); \qquad v(0) = v_0$$

- Under certain conditions can prove that $v \to u$.

Inverse Problems

- Many problems in mathematics seek to predict the effects of a given set of causal factors
- Inverse problems seek to determine the most likely causal factors for a set of observations
- Example: given (limited) observations of a fluid, determine its viscosity

Nudging, continued

Continuing from before, suppose our system is

$$u_t + F(u) + \lambda G(u) = 0,$$

where λ is a parameter. We add the coupled system

$$v_t + F(v) + \tilde{\lambda} G(v) = \mu P(u - v); \qquad v(0) = v_0$$

Goal: update $\tilde{\lambda}$ so that $\tilde{\lambda} \to \lambda$ over time.

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Previous Work

- Basics: Azouani, Olson, and Titi
- Recent applications:
  - Kuramoto-Sivashinsky equation - Pachev et. al
  - Lorenz equations - Carlson et. al.
  - 2D Naiver-Stokes equations - Martinez

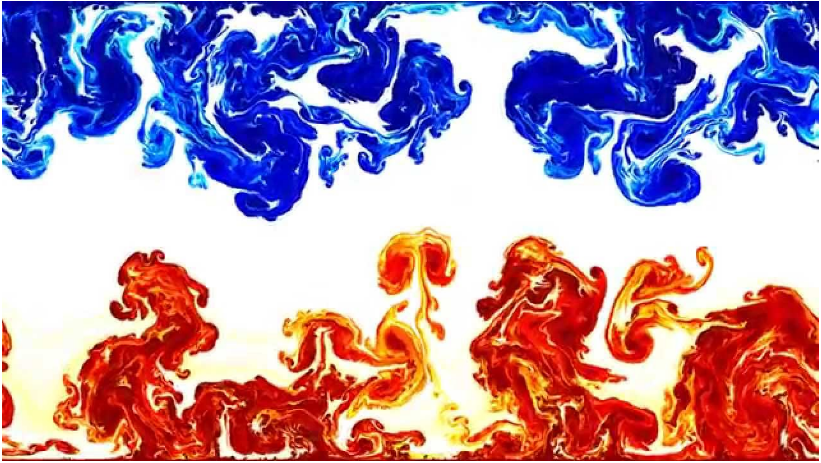
Rayleigh-Bénard Convection

- A fluid which is heated from below and cooled from above
- Modeled by the *Boussinesq Approximation*:

$$u_t + u \cdot \nabla u + \nabla p = \mathrm{Pr}\Delta u + \mathrm{Pr}\mathrm{Ra}\theta\hat{z}$$
$$\nabla \cdot u = 0$$
$$\theta_t + u \cdot \nabla \theta = \Delta \theta$$

- Parameters of interest: Pr and Ra

Rayleigh-Bénard Convection, continued



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Point-in-Time Updates

- Formal computations show that choosing

$$\tilde{Pr}\,\alpha_i + \tilde{PrRa}\,\beta_i = \gamma_i,$$

(for some constants $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$) leads to

$$\frac{1}{2}\frac{d}{dt}\|P(u-v)\|^2 = -\mu\|P(u-v)\|^2.$$

- Hypothesis: This will make $u \to v$ and the parameters converge as well.

Point-in-Time Updates, continued

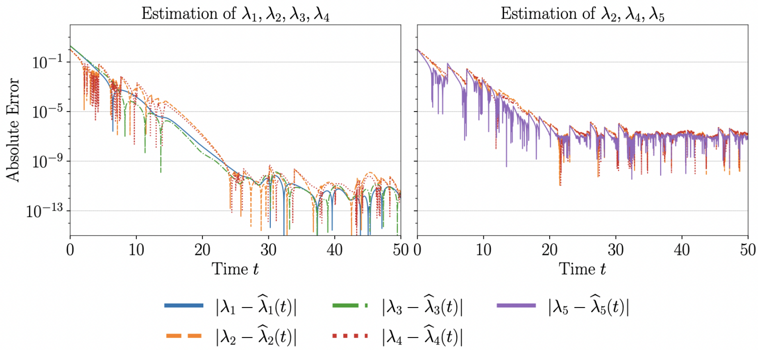

Fig. 3: Convergence of multi-parameter estimation with respect to time for the four parameters corresponding to linear terms in the generalized KSE (3.3b) (left) and for the three parameters corresponding to terms in the original KSE, including the nonlinear coefficient (right).

Linearized Updates

- Add a nudging term to the temperature equation
- Neglect terms quadratic in the error
- Let w be the error in velocity field and ω be the error in the temperature field. We can make the update

$$Ra \approx \tilde{Ra} - \mu \frac{\| I_h(w) \|_2^2}{\langle I_h(\theta)_x, I_h(w) \rangle},$$

$$Pr^{-1} \approx \tilde{Pr}^{-1} - \mu \frac{\| I_h(\omega) \|_2^2}{\langle I_h(\Delta\tilde{\theta}), I_h(\omega) \rangle}.$$

Linearized Updates, continued

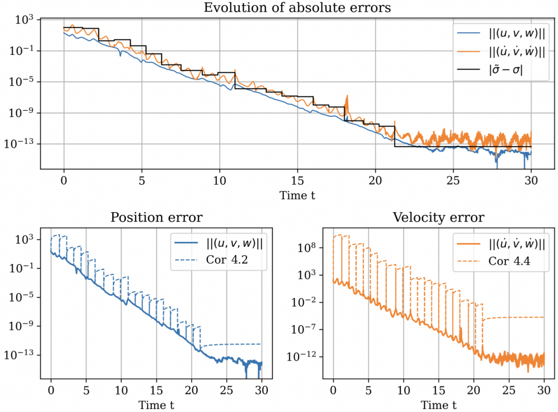

FIGURE 1. The parameter learning algorithm is applied to the true parameters $\sigma = 10$, $\rho = 28$, $\beta = 8/3$ with ρ, β known and σ recovered from continuous observations in $x(t)$. The initial guess is $\sigma_0 = \sigma + 100$ and the algorithm parameters were set to $\mu_1 = 500$, $T_R = 1$. The analytically derived upper bounds on position and velocity error from Corollary 4.2 and Corollary 4.4 are shown to hold remarkably well.

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Current Work

- Use dedalus to computationally verify
- See where the algorithm works best
- Change if necessary

Future Work

- Compare the two approaches
- Work on an analytic proof of convergence

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- Other authors of the papers described
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- Sources:

  - https://www.youtube.com/watch?v=OM0l2YPVMf8
  - Pachev, Whitehead, McQuarrie. Concurrent multi-parameter learning demonstrated on the Kuramoto-Sivashinsky equation. arXiv:2106.06069
  - Carlson, Hudson, Larios, Martinez, Ng, Whitehead. Dynamically learning the parameters of a chaotic system using partial observations. arXiv:2108.08354
  - Martinez. Convergence Analysis of a Viscosity Parameter Recovery Algorithm for the 2D Navier-Stokes Equations. arXiv:2110.11568
  - Azouani, Olson, Titi. Continuous Data Assimilation Using General Interpolant Observables. arXiv:1304.0997